MODELLING WITH EXPONENTIAL FUNCTIONS
WORKSHEET 2

QUESTION 1
The spread of the flu virus in a town of 5000 people is modelled by the rule
$y = \dfrac{5000}{1 + 4999e^{-0.8t}}$, $t \geq 0$, where y is the number of people infected after t days.

(a) How many people are infected with the flu after 3 days?

(b) How long does it take for half the town to become infected? State your answer to the nearest hour.

QUESTION 2

A certain lake is stocked with 1000 fish. The population is growing according to the logistics curve: $P = \dfrac{10,000}{1+9e^{-t/5}}$ where t is measured in months since the lake was initially stocked.

(a) Find the population in 8 months. State your answer to the nearest fish.

(b) After how many months will the fish population be 2000? State your answer correct to 3 decimal places.

(c) Is there a maximum possible fish population that the lake can sustain? What graphical feature is this on the graph?

QUESTION 3

A colony of bacteria increases according to the rule $N(t) = N_0 e^{kt}$, where N_0 is the initial number of bacteria present.

(a) If the number of bacteria doubles in 3 hours, find the function that gives the number of cells in the colony.

(b) How long will it take for the size of the colony to triple? State your answer correct to the nearest minute.

QUESTION 4

The function $L(t) = A(1 - e^{-kt})$ is used to measure the amount, L, learned, at time t minutes. A represents the amount to be learned and the number k measures the rate of learning.

Suppose that a student has an amount A of 200 words to learn. A psychologist determines that the student has learned 20 words after 5 minutes.

(a) Determine the rate of learning, k, correct to 3 decimal places.

(b) How many words will the student have learned after 10 minutes? State your answer correct to the nearest word.

(c) How long does it take for the student to learn 180 words? State your answer correct to the nearest minute.

(d) Sketch the graph of $L(t)$ labelling the coordinates of axial intercepts and the equation of any asymptotes.

QUESTION 5

The population of crickets was studied in two locations, one on each end of a small isle in the Pacific Ocean.

The first colony was found to be modelled by $P(t) = 50e^{0.1t}$, $t \geq 0$, where P is the population of crickets and t is the number of days from the start of the study.

(a) (i) What was the population of crickets when the study began? State your answer to the nearest cricket.

(ii) What was the population of crickets 12 days after the study began? State your answer to the nearest cricket.

(iii) When will the population of crickets first exceed 75?

(iv) How is the population of crickets changing?

The population of crickets in the second colony was found to be approximated by the model $Q(t) = 500 - 450e^{-0.1t}$, $t \geq 0$.

(b) Sketch the graph of the population Q against time labelling the coordinates of the axial intercepts, endpoints and the equations of any asymptotes.

(c) (i) By solving an appropriate equation, show that when the two colonies have the same number of crickets, then $k^2 - 10k + 9 = 0$ where $k = e^{0.1t}$.

(ii) Hence find the time when the two cricket colonies have equal numbers.

(d) (i) Explain briefly why an inverse function Q^{-1} exists.

(ii) Find $Q^{-1}(t)$.

SOLUTIONS

QUESTION 1

(a)

Find y when $t=3$

$$y = \frac{5000}{1+4999e^{-0.8\times 3}} = 11.0011$$

$$= 11 \text{ people}$$

(b)

Find t when $y = 2500$

$$2500 = \frac{5000}{1+4999e^{-0.8t}}$$

$$1+4999e^{-0.8t} = \frac{5000}{2500}$$

$$1+4999e^{-0.8t} = 2$$

$$4999e^{-0.8t} = 1$$

$$e^{-0.8t} = 0.0002$$

$$\log_e e^{-0.8t} = \log_e 0.0002$$

$$-0.8t = \log_e 0.0002$$

$$t = -\frac{\log_e 0.0002}{0.8} = 10.6465 \text{ days}$$

$$= 10 \text{ days } 15.516 \text{ hrs}$$

$$= 10 \text{ days } 16 \text{ hrs}$$

QUESTION 2

(a) $P(8) = \dfrac{10{,}000}{1+9e^{-8/5}} \approx 3550$ fish

(b) $2000 = \dfrac{10{,}000}{1+9e^{-t/5}} \Rightarrow 1+9e^{-t/5} = 5 \Rightarrow e^{-t/5} = 4/9 \Rightarrow -\dfrac{t}{5} = \ln(4/9) \Rightarrow t = -5\ln(4/9)$
so $t \approx 4.055$ months

(c) The maximum possible fish population is 10,000 fish ($t \to \infty$) which is the horizontal asymptote on the graph.

QUESTION 3

(a)

$N = N_0 e^{kt}$

when $t = 0$, $N = N_0$

when $t = 3$, $N = 2N_0$

$$N(3) = N_0 e^{3k} = 2N_0$$

$$e^{3k} = 2$$

$$\log_e e^{3k} = \log_e 2$$

$$3k = \log_e 2$$

$$k = \frac{1}{3}\log_e 2$$

$$\therefore N = N_0 e^{\left(\frac{1}{3}\log_e 2\right)t}$$

(b)

Find t when $N = 3N_0$

$$N = N_0 e^{(1/3 \log_e 2)t}$$

$$3N_0 = N_0 e^{(1/3 \log_e 2)t}$$

$$3 = e^{(1/3 \log_e 2)t}$$

$$(1/3 \log_e 2)t = \log_e 3$$

$$t = \frac{\log_e 3}{1/3 \log_e 2}$$

$$t = \frac{3\log_e 3}{\log_e 2} = \frac{\log_e 27}{\log_e 2}$$

$$= 4.75489$$

$$= 4 \text{ hrs } 45 \text{ minutes}$$

QUESTION 4

(a)

$A = 200$

$L = 20 \qquad L = A(1 - e^{-kt})$

$t = 5$

$\therefore \; 20 = 200(1 - e^{-5k})$

$0.1 = 1 - e^{-5k}$

$-0.9 = -e^{-5k}$

$\therefore \; 0.9 = e^{-5k}$

$\log_e 0.9 = \log_e e^{-5k}$

$\therefore -5k = \log_e 0.9$

$k = 0.021072 = 0.021$

$\therefore \; L = 200(1 - e^{-0.021t})$

(b)

Find L when $t = 10$

$L = 200(1 - e^{-0.21}) = 37.8832$

$= 38$ words

(c)

Find t when $L = 180$

$180 = 200(1 - e^{-0.021t})$

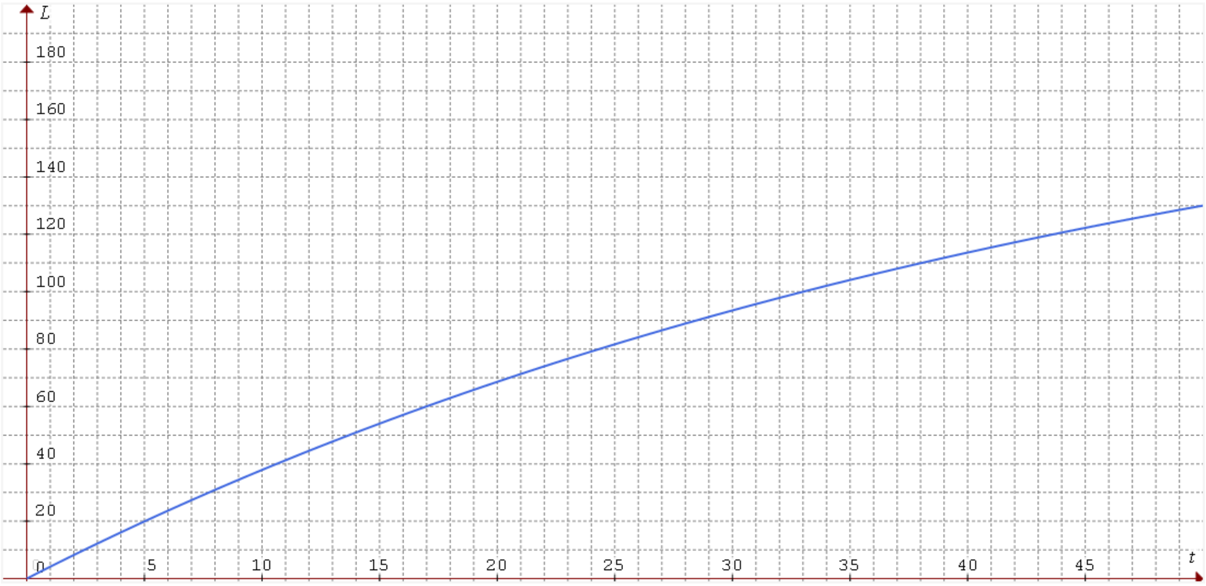
$0.9 = 1 - e^{-0.021t}$

$-0.1 = -e^{-0.021t}$

$-0.021t = \log_e 0.1$

$t = 109.647 = 110$ minutes

(d)



QUESTION 5

(a) (i)

$P = 50e^{0.1t}$

Find P when $t = 0$

$P = 50e^{0} = 50$

(ii)

Find P when $t = 12$

$P = 50e^{1.2} = 166.006$

$= 166$

(iii)

Find t when $P = 75$

$P = 50e^{0.1t}$

$75 = 50e^{0.1t}$

$1.5 = e^{0.1t}$

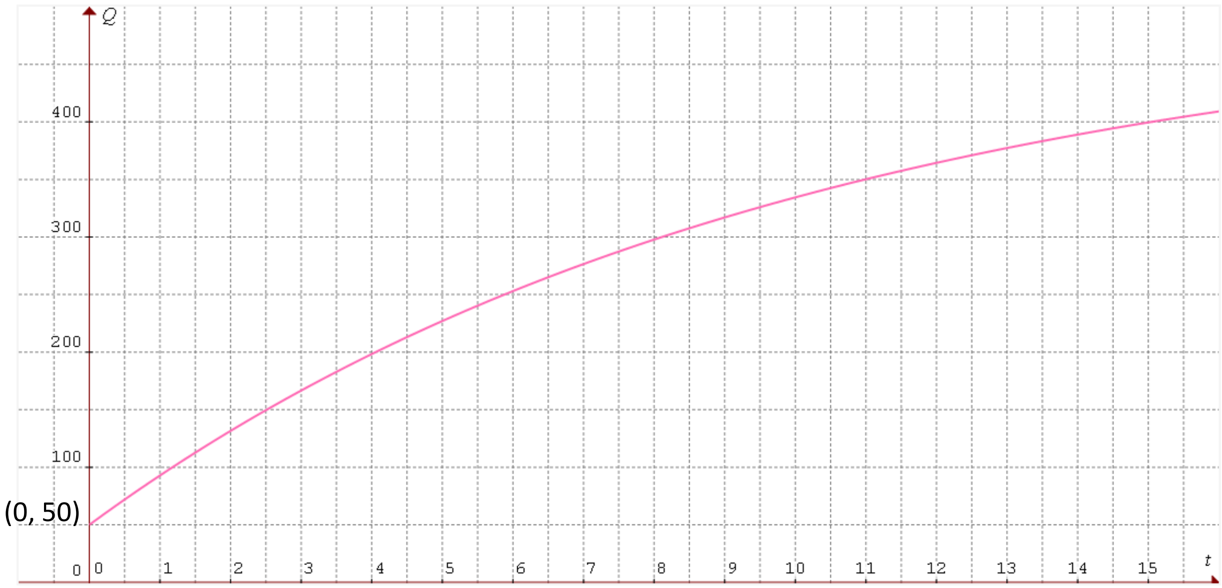
$\log_e 1.5 = \log_e e^{0.1t}$

$0.1t = \log_e 1.5$

$t = 10\log_e 1.5$ days

(iv) The population of crickets is increasing.

(b)



(c) (i)

$$500-450e^{-0.1t} = 50e^{0.1t}$$

$$500-450e^{-0.1t} - 50e^{0.1t} = 0$$

$$50-45e^{-0.1t} - 5e^{0.1t} = 0$$

$$50-\frac{45}{e^{0.1t}} - 5e^{0.1t} = 0$$

Let $k=e^{0.1t}$

$$50-\frac{45}{k} - 5k = 0$$

$$\frac{50k-45-5k^2}{k} = 0$$

$$-5k^2+50k-45 = 0$$

$$k^2-10k+9 = 0$$

(ii)

$$k^2-10k+9 = 0$$

$$(k-9)(k-1) = 0$$

$$k = 1, 9$$

$e^{0.1t} = 1$ $\qquad$ $e^{0.1t} = 9$

$e^{0.1t} = e^0$ $\qquad$ $\log_e e^{0.1t} = \log_e 9$

$t = 0$ $\qquad$ $0.1t = \log_e 9$

$t = 10\log_e 9$ days

d. (i) As $Q(t)$ is a one to one function.

(ii)

$$y = 500 - 450e^{-0.1t}$$

Interchanging y and t

$$t = 500 - 450e^{-0.1y}$$

$$t - 500 = -450e^{-0.1y}$$

$$\frac{t-500}{-450} = e^{-0.1y}$$

$$\log_e\left(\frac{500-t}{450}\right) = \log_e e^{-0.1y}$$

$$\therefore -0.1y = \log_e\left(\frac{500-t}{450}\right)$$

$$y = -10\log_e\left(\frac{500-t}{450}\right)$$

$$\therefore Q^{-1}(t) = -10\log_e\left(\frac{500-t}{450}\right),\ t \in [50, \infty)$$